Ministero dell'Istruzione, dell'Università, della Ricerca
Ufficio Scolastico Regionale Per Il Lazio
Liceo Scientifico Statale
"TALETE"

Modulo 3 MiniCLIL

Incontro 8: Sequences and Series

L'UdA della giornata ha lo scopo di introdurre e approfondire il tema matematico delle serie numeriche. Le somme di elementi di una successione numerica, appartenenti a spazi vettoriali topologici, costituiranno il materiale di esercitazione e confronto all'interno dei gruppi di lavoro che, animati da interventi di 'peer to peer' si eserciteranno mediante CLIL applicativo.

Lezione

Il materiale di esercitazione della giornata è tratto dal testo *'Sequences and Series', OCR (A) Mathematics (Pure)* utilizzato nelle scuole superiori inglesi per la preparazione degli A levels di matematica. Il materiale riprodotto e in allegato segue i termini di copyright inglesi ed è stato gentilmente concesso dalla Yiewsley Library, West Drayton, Londra.

Chapter 9 Sequences and series

About this topic

You meet sequences and series in everyday life; they often provide the patterns of events around us. The two most common types of sequence are arithmetic and geometric and they have a variety of real-life applications including in banking, medicine and manufacturing.

Before you start, remember

- number patterns from GCSE
- how to solve equations.

Definitions and notation

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Key facts

1 A **sequence** is an ordered set of numbers $a_1, a_2, \ldots\ldots\ldots a_k, \ldots\ldots a_n$

2 Sequences can be finite or infinite
- a **finite sequence** has a first term and a last term
- an **infinite sequence** is a sequence which continues forever.

3 There are two common ways of defining a sequence
- as a position-to-term or **deductive** formula like $a_k = k^2 - 1$ for $k = 1, 2, 3, \ldots$
- as a term-to-term or **inductive** rule like $a_1 = 4$; $a_{k+1} = 2a_k + 3$.

4 In an **arithmetic sequence** (also called an **arithmetic progression**) the difference between two consecutive terms is constant. So $a_{k+1} = a_k + d$, where d is a fixed number called the **common difference**.

5 In a **geometric sequence** (also called a **geometric progression**) the ratio between two consecutive terms is constant. So $a_{k+1} = ra_k$, where r is a fixed number called the **common ratio**.

6 In an **increasing sequence** each term is greater than the term before. In a **decreasing sequence** each term is less than the term before.

7 In a **periodic sequence** $a_{k+p} = a_k$ for a fixed integer p, called the **period**.

8 In a **convergent sequence** the differences between successive terms is smaller each time. A convergent sequence tends to a **limit** as $n \to \infty$ and $a_k \to a_{k+1}$.

9 A sequence which is not convergent is said to be a **divergent** sequence. The terms may tend to $+\infty$ or $-\infty$ or to no particular value.

10 A **series** is the sum of the terms of a sequence. S_n denotes the sum to n terms of the series.

$$S_n = \sum_{k=1}^{k=n} a_k = a_1 + a_2 + \ldots + a_n$$

e.g. 1, 4, 7, 10...

Each term can be worked out from its position in the sequence.

This is also called a **recurrence relation**.

To find the next term you add d to the previous term (d could be positive or negative).

To find the next term you multiply the previous term by r (r could be positive or negative).

e.g. 1, 5, 25, 1, 5, 25, 1, 5, 25... has period 3 and $a_1 = a_4 = a_7 = \ldots$

e.g. $1, -\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \ldots$ converges to zero.

The terms get progressively closer to a fixed number called the limit of the sequence.

$-1, -4, -7, -10, \ldots$

e.g. 1, 2, 4, 8, 16, ...

e.g.

$$S_5 = \sum_{k=1}^{k=5} k^2 = 1 + 4 + 9 + 16 + 25 = 55.$$

Worked examples

1 Recognising types of sequence

Describe the sequence 2, 4, 6, 8, 2, 4, 6, 8, 2, 4, 6, 8, ...

Solution

This sequence repeats itself every 4th term so it is periodic with period 4.

It is infinite and divergent.

> There is no last term and the terms do not converge.

2 Finding the terms in a sequence from a deductive formula

A sequence is defined by $a_k = k^3 - k^2 + 1$ for $k = 1, 2, 3, \ldots$

Write down the first four terms of the sequence.

Solution

$a_k = k^3 - k^2 + 1$

When $k = 1$, $a_1 = 1^3 - 1^2 + 1 = 1$

When $k = 2$, $a_2 = 2^3 - 2^2 + 1 = 8 - 4 + 1 = 5$

When $k = 3$, $a_3 = 3^3 - 3^2 + 1 = 27 - 9 + 1 = 19$

When $k = 4$, $a_4 = 4^3 - 4^2 + 1 = 64 - 16 + 1 = 49$

3 Using an inductive formula or recurrence relation

A sequence is defined by $a_1 = 2$; $a_{k+1} = 0.8a_k + 3$.

i Calculate the value of a_3.

ii What is the smallest value of n for which $a_n \geqslant 10$?

Solution

i $a_1 = 2$,

$a_2 = 0.8a_1 + 3 = 0.8 \times 2 + 3 = 4.6$,

$a_3 = 0.8a_2 + 3 = 0.8 \times 4.6 + 3 = 6.68$

> Notice that the difference between successive terms is growing smaller each time. So the sequence is convergent. It converges to 15 (use your calculator to check this).

ii *To find the first term that is over 10, find the terms one by one*

$a_4 = 0.8a_3 + 3 = 0.8 \times 6.68 + 3 = 8.344$,

$a_5 = 0.8a_4 + 3 = 0.8 \times 8.344 + 3 = 9.6752$,

$a_6 = 0.8a_5 + 3 = 0.8 \times 9.6752 + 3 = 10.74016$, so $a_6 \geqslant 10$

So 6 is the smallest value of n for which $a_n \geqslant 10$.

4 Solving problems involving sequences

A sequence is defined by $a_{k+1} = pa_k + q$ where $a_1 = 48$.

Given that $a_2 = 20$ and $a_3 = 13$, find the values of p and q.

Solution

$a_2 = 20 \Rightarrow 48p + q = 20$ (1)

$a_3 = 13 \Rightarrow 20p + q = 13$ (2)

> There are 2 unknowns so you need to form 2 equations.

Subtracting (2) from (1) gives $28p = 7 \quad \Rightarrow p = \frac{1}{4}$.

Substituting $p = \frac{1}{4}$ in (1) $\Rightarrow 48 \times \frac{1}{4} + q = 20 \Rightarrow q = 8$.

So $p = \frac{1}{4}$, $q = 8$.

5 Using series

The sum of n terms of a series is given by $S_n = \frac{n^2(n+1)^2}{4}$.

i Write down the first four terms of the series.

ii Find an expression for the nth term of the series.

Solution

i $S_1 = a_1 = \frac{1^2 \times 2^2}{4} = \frac{4}{4} = 1 \quad \Rightarrow a_1 = 1$

$S_2 = a_1 + a_2 = \frac{2^2 \times 3^2}{4} = 9 \quad \Rightarrow a_2 = 8$

$S_3 = a_1 + a_2 + a_3 = \frac{3^2 \times 4^2}{4} = 36 \quad \Rightarrow a_3 = 27$

$S_4 = a_1 + a_2 + a_3 + a_4 = \frac{4^2 \times 5^2}{4} = 100 \Rightarrow a_4 = 64$

Notice that these are all cube numbers. You will prove this result in part ii.

So the series is $1 + 8 + 27 + 64...$

ii In general, $a_n = S_n - S_{n-1}$.

For example, $a_2 = S_2 - S_1$ and $a_3 = S_3 - S_2$.

$$a_n = S_n - S_{n-1} = \frac{n^2(n+1)^2}{4} - \frac{(n-1)^2 n^2}{4}$$

$$= \frac{n^2(n^2+2n+1)-(n^2-2n+1)n^2}{4}$$

$$= \frac{n^4+2n^3+n^2-(n^4-2n^3+n^2)}{4}$$

$$= \frac{4n^3}{4}$$

So the nth term of the series is n^3.

Check the first four terms of the series to see that this is correct.

6 Using sigma notation

A sequence is defined by $a_k = (k+1)2^k$.

Write out the series $\sum_{k=2}^{5} a_k$ without simplifying the terms.

Solution

Substituting $k = 2$ into $(k+1)2^k \Rightarrow a_2 = 3 \times 2^2$

Substituting $k = 3$ into $(k+1)2^k \Rightarrow a_3 = 4 \times 2^3$

Substituting $k = 4$ into $(k+1)2^k \Rightarrow a_4 = 5 \times 2^4$

Substituting $k = 5$ into $(k+1)2^k \Rightarrow a_5 = 6 \times 2^5$

So $\sum_{k=2}^{5}(k+1)2^k = 3 \times 2^2 + 4 \times 2^3 + 5 \times 2^4 + 6 \times 2^5$

Test yourself

TESTED

1 Which of the following is the best description of the sequence whose nth term is $\cos(n \times 60°)$?

A divergent and geometric

B periodic with period 6

C both divergent and periodic with period 6

D convergent and geometric

2 The sum of n terms of a series is given by $S_n = \frac{n}{n+1}$.

Which of the following is the correct series?

A $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}...$ B $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}...$ C $\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{72}...$ D $\frac{1}{2}+\frac{7}{6}+\frac{17}{12}+\frac{31}{20}...$

3 Which one of the following statements is true?

A $\sum_{r=3}^{7} r^2 = 140$

B The sequence 1, −1, 1, −1, 1... converges

C The sequence 1, 3, 5, 7,... is defined by $a_{k+1} = a_k + 2$

D The sum to n terms of the series $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}...$ is $2-\left(\frac{1}{2}\right)^{n-1}$

4 Which of the following series is the same as $1 - x + 2x^2 - 3x^3 + ...$?

A $\sum_{r=1}^{\infty}(-1)^r rx^r$ B $1+\sum_{r=1}^{\infty}(-1)^r rx^r$ C $1-\sum_{r=1}^{\infty}(-1)^r rx^r$ D $\sum_{r=1}^{\infty}(-1)^{r+1} rx^r$

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Exam-style question

Jan is being treated with a special drug.

At 0100 hours she is given 120 units of the drug.

Each hour the amount in her body reduces by 10 units.

At 0600, 1100, and 1600 she is given subsequent doses, in each case enough to bring the amount in her body up to 120 units.

The amounts in her body every hour are denoted by a_1 at 0100, a_2 at 0200 and so on.

i Write down the sequence $a_1, a_2, ..., a_{20}$.

ii Describe this sequence.

Jan is not given any more of the drug, (she is recovering well).

iii Write down the value of a_{21}.

iv What is the mean amount of drug in Jan's body from 0100 to 2100?

v What is the total amount of the drug given to Jan?

vi When is there no drug left in Jan's body?

Short answers on pages 220–221

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Sequences and series

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Key facts

1 In an **arithmetic** sequence (progression) with first term a, common difference d and n terms:
- the kth term is given by $a_k = a+(k-1)d$
- the last term, $l = a_n = a+(n-1)d$
- the sum of n terms is $S_n = \frac{1}{2}n(a+l) = \frac{1}{2}n[2a+(n-1)d]$.

2 In a **geometric** sequence (progression) with first term a, common ratio r and n terms:
- the kth term is given by $a_k = ar^{k-1}$
- the last term, $a_n = ar^{n-1}$
- the sum of n terms is $S_n = \frac{a(r^n-1)}{(r-1)} = \frac{a(1-r^n)}{(1-r)}$
- for an **infinite** geometric series to converge the common ratio must be between −1 and 1, so $-1<r<1$ which is sometimes written $|r|<1$
- the **sum to infinity** of a convergent G.P. is $S = S_\infty = \frac{a}{1-r}$.

Worked examples

1 Finding arithmetic sequences

The fourth term of an arithmetic sequence is 13 and the seventh term is 19.

i Find the first term. ii Find the nth term.

Solution

i Using $a_k = a+(k-1)d$ gives
4th term $a_4 = a+3d = 13$
7th term $a_7 = a+6d = 19$
By subtraction $3d = 6$
Hence $d = 2$ and $a = 7$.
So the first term is 7.

ii The nth term is $7+(n-1)\times 2 = 5+2n$

Find $a_7 - a_4$.

Use $l = a_n = a+(n-1)d$.

Check: when $n = 7$ then $5+2\times 7 = 19$ ✓

2 Solving problems involving arithmetic sequences

i Show that the series whose kth term is given by $a_k = 3k+1$ is an arithmetic series.

ii Find the 20th term and the sum to 20 terms.

Solution

i By substitution
$a_1 = 3\times 1+1 = 4$
$a_2 = 3\times 2+1 = 7$
$a_3 = 3\times 3+1 = 10$
In general, $(k+1)$th term, $a_{k+1} = 3(k+1)+1 = 3k+4$
kth term: $a_k = 3k+1$
Difference: $a_{k+1} - a_k = 3$
So the sequence is an arithmetic progression with first term 4 and common difference 3.

ii The 20th term is $3\times 20+1 = 61$.
Using $S_n = \frac{1}{2}n(a+l)$,
the sum to 20 terms is $\frac{1}{2}\times 20(4+61) = 650$.

You can see that so far there is a common difference of 3. Now you need to prove that this is true for any pair of terms.

$(3k+4)-(3k+1) = 4-1 = 3$.

3 Finding geometric sequences

A geometric sequence has second term 3 and fifth term 24.

i Find the first term and the common ratio.

ii Find the 8th term and the sum to 8 terms.

Solution

i Using $a_k = ar^{k-1}$ gives:

2nd term = 3 $= ar^2 \Rightarrow ar = 3$,

5th term = $24 = ar^{5-1} \Rightarrow ar^4 = 24$

So $\frac{ar^4}{ar} = \frac{24}{3}$

$\Rightarrow r^3 = 8$

Hence $r = 2$ and $a = 1.5$

ii 8th term: $a_8 = ar^{8-1} = 1.5 \times 2^7 = 192$

Sum to 8 terms: $S_8 = \frac{1.5 \times (2^8 - 1)}{2-1} = 382.5$

This is a very useful technique: dividing one equation by the other cancels a.

Using $ar = 3$.

Using $S_n = \frac{a(r^n - 1)}{(r-1)}$.

4 Finding the sum to infinity

i Show that the geometric series

$5 + \frac{5}{2} + \frac{5}{4} + \frac{5}{8}......$

has a sum to infinity.

ii Find the sum to infinity.

Solution

i The first term is $a = 5$.

Each successive term is half of its predecessor so $r = \frac{1}{2}$.

Since $-1 < r < 1$, the geometric sequence is convergent and has a sum to infinity.

ii Sum to infinity $S = \frac{a}{1-r} = \frac{5}{1-\frac{1}{2}} = \frac{5}{\frac{1}{2}} = 10$.

Remember that a geometric series can only have a sum to infinity if it converges, so you need to show the common ratio is between −1 and 1.

5 Solving problems involving geometric series

i State the common ratio of the geometric series

$3 + \frac{3x}{2} + \frac{3x^2}{4} + \frac{3x^3}{8}......$

ii State the restrictions on x for the series to have a sum to infinity.

iii State the sum to infinity in terms of x.

iv Find x if the sum to infinity is 15.

Solution

i The first term is $a = 3$.

The common ratio is $r = \frac{x}{2}$

ii For sum to infinity to exist $-1 < r < 1$

so, in this case, $-1 < \frac{x}{2} < 1$.

Multiplying through by 2 gives $-2 < x < 2$.

iii $S = \frac{a}{1-r} = \frac{3}{1-\frac{x}{2}} = \frac{6}{2(1-\frac{x}{2})} = \frac{6}{2-x}$

iv Given that $\frac{6}{2-x} = 15$

$\Rightarrow \quad 6 = 15(2-x)$

$= 30 - 15x$

$\Rightarrow \quad 15x = 24$ so $x = 1.6$ or $1\frac{3}{5}$.

The terms in the sequence involve x, so the common ratio is in terms of x.

The sequence must converge in order to have a sum to infinity.

Multiply both the numerator and denominator by 2 to clear the fraction, $\frac{x}{2}$, in the denominator.

Notice that this satisfies $-2 < x < 2$.

Test yourself

TESTED

1 The numbers p, 4, q, form a geometric sequence.
Which of the following values of p and q are possible?

A $p = 0, q = 8$ B $p = 0, q = 16$ C $p = 1, q = 16$ D $p = 6, q = 2$

2 The first three terms of an arithmetic sequence are –3, 2 and 7.
What is the sum of the first 12 terms?

A 52 B 294 C 324 D 648

3 The 2nd term of an arithmetic sequence is 7 and the 6th term is –5.
Three of the following statements are false and one is true. Which one is true?

A The common difference is 3
B The first term is 13
C $a_k < -10$ if $k \geqslant 8$
D $\sum_{k=2}^{8} a_k = -4$

4 In which of the following cases does the series $1 - 2x + 4x^2 - 8x^3 + \ldots$ have the given sum to infinity, S?

A $x = 3,\ S = \frac{1}{7}$
B $x = -\frac{1}{5},\ S = \frac{5}{7}$
C $x = \frac{1}{3},\ S = \frac{5}{3}$
D $x = \frac{1}{4},\ S = \frac{2}{3}$

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Exam-style question

i On Ian's first birthday his parents put £25 into a money box. On his second birthday they give him £28, on his third birthday £31 and so on.
 a How much money does he receive on his 18th birthday?
 b How much money is in his money box at the end of his 18th birthday if none has been spent?

ii On Ian's eleventh birthday his grandparents put £200 into a bank account for him.
Compound interest at 7% is added to this account every following year on his birthday.
His grandparents add another £200 on each birthday.
How much is in his account the day after Ian is eighteen?

Short answers on page 221

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